Algorithms and Data Structures

Searching in Lists

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Topic of Next Lessons

- Search: Given a (sorted or unsorted) list A with |A|=n elements (integers). Check whether a given value c is contained in A or not
  - Search returns true or false
  - If A is sorted, we can exploit transitivity
  - Fundamental problem with a zillion applications
- Select: Given an unsorted list A with |A|=n elements (integers). Return the i'th largest element of A.
  - Returns an element of A
  - The sorted case is trivial – return A[i]
  - Interesting problem (especially for median) with many applications
  - [Interesting proof]

Content of this Lecture

- Searching in Unsorted Lists
- Searching in Sorted Lists
- Selecting in Unsorted Lists

Searching in an Unsorted List

- No magic
- Compare c to every element of A
- Worst case ($c \notin A$): O(n)
- Average case ($c \in A$)
  - If c is at position i, we require i tests
  - All positions are equally likely: probability 1/n
  - This gives

$$\frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n} * \frac{n^2+n}{2} = \frac{n+1}{2} = O(n)$$

```
A: unsorted_int_array;
c: int;
for i := 1.. |A| do
  if A[i]=c then
    return true;
  end if;
end for;
return false;
```

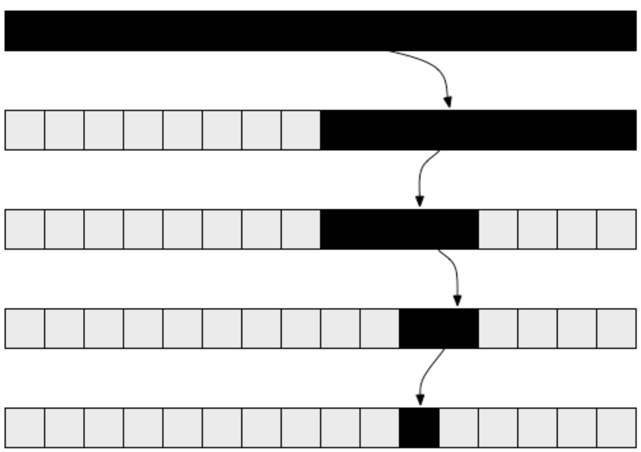
Content of this Lecture

- Searching in Unsorted Lists
- Searching in Sorted Lists
  - Binary Search
  - Fibonacci Search
  - Interpolation Search
- Selecting in Unsorted Lists

Binary Search (binsearch)

- If A is sorted, we can be much faster
- Binsearch: Exploit transitivity

```
func bool binsearch(A: sorted_array;
                          c,l,r : int) {
  If l>r then
    return false;
  end if;
  m := (l+r) div 2;
  If c<A[m] then
    return binsearch(A, c, l, m-1);
  else if c>A[m] then
    return binsearch(A, c, m+1, r);
  else
    return true;
  end if;
}
```



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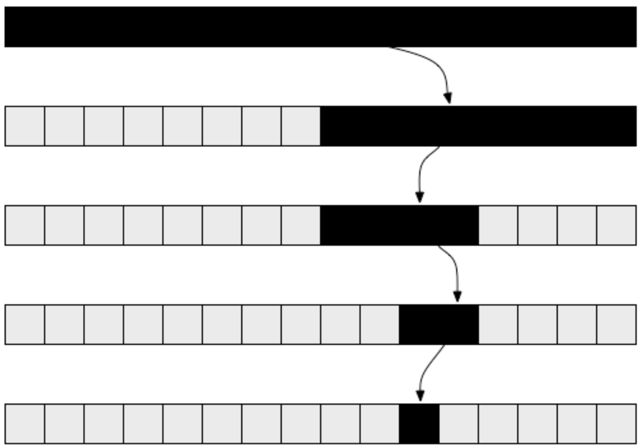
Iterative Binsearch

- Binsearch uses only end-recursion
- Equivalent iterative program
  - No call stack
  - We don't need old values for l,r
  - O(1) additional space

```
A: sorted_int_array;
c: int;
l := 1;
r := |A|;
while l≤r do
  m := (l+r) div 2;
  if c<A[m] then
    r := m-1;
  else if c>A[m] then
    l := m+1;
  else
    return true;
end while,
return false;
```

Complexity of Binsearch

- In every call to binsearch (or every while-loop), we only do constant work
- With every call, we reduce the size of sub-array by 50%
  - We call binsearch once with n, with n/2, with n/4, …
- Binsearch has worst-case complexity O(log(n))
- Average case only marginally better
  - Chances to "hit" target in the middle of an interval are low in most cases
  - See Ottmann/Widmayer



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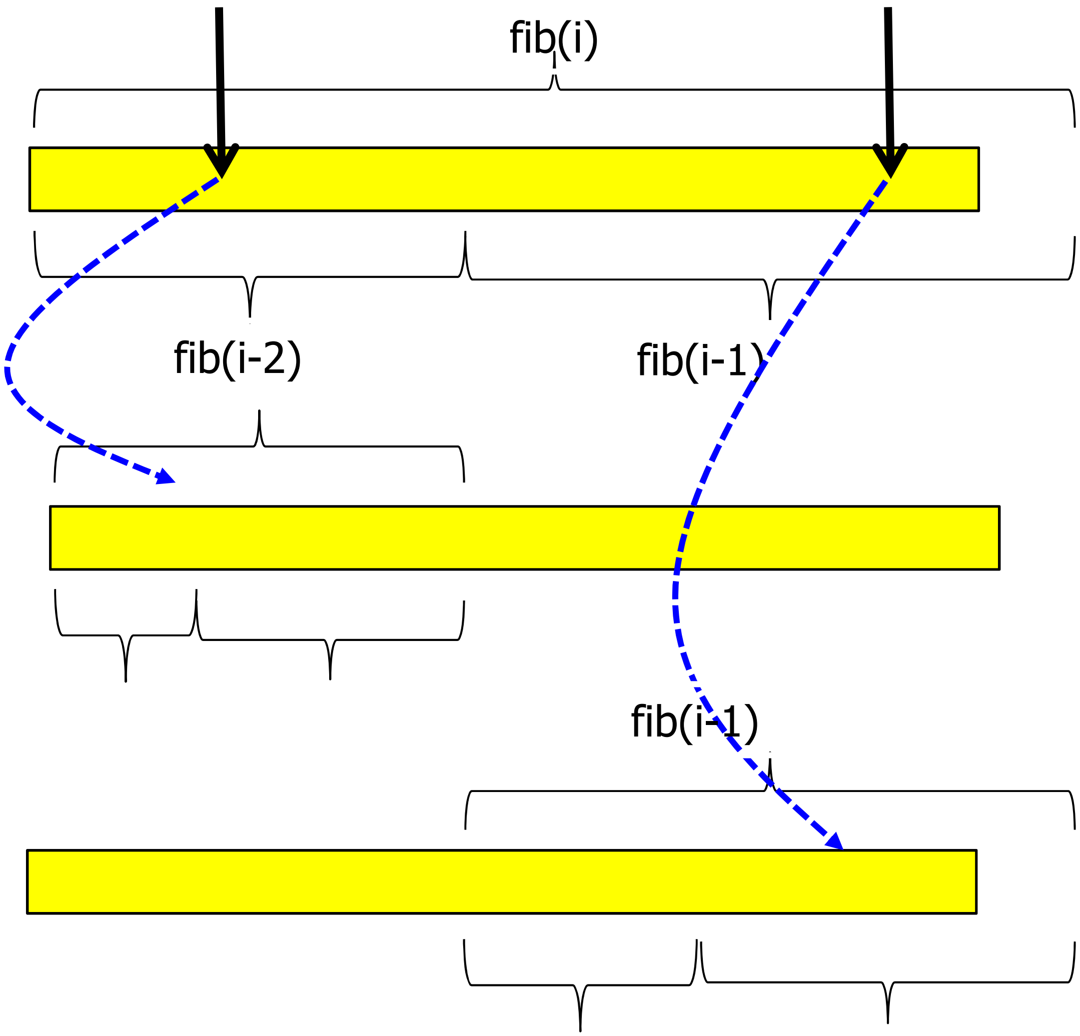
Searching without Divisions

- If we want to be ultra-fast, we should use only simple arithmetic operations
- Fibonacci search: O(log(n)) without division/multiplication
  - Note: Bin-search usually uses bit shift (div 2) – very fast
  - Fibonacci search also has slightly better access locality (cache)
  - Also interesting: O(log(n)) without the "always 50%" trick
- Recall Fibonacci numbers
  - fib(1)=fib(2)=1; fib(i)=fib(i-1)+fib(i-2)
  - 1, 1, 2, 3, 5, 8, 13, 21, 34, …
  - Thus, fib(i-2) is roughly 1/3, fib(i-1) roughly 2/3 of fib(i)

Fibonacci Search: Idea



- Let fib(i) be the smallest fib-number >|A|
- If A[fib(i-2)]=c: stop
- Otherwise, continue searching in [1 … fib(i-2)] or [fib(i-2)+1 … n]
- Beware out-of-range part A[n+1…fib(i)]
- No divisions

Algorithm (assume |A|=fib(n)-1)

- 3-6: Search at A[fib(i-2)]
  - With fib1, fib2 we can compute all all other fib's
  - fib(i)=fib(i-1)+fib(i-2)
  - fib(i-1)=fib(i-2)+fib(i-3)
  - …
- 7-24: Break A at descending Fibonacci numbers
- After each comparison, update fib1 and fib2

```
1.  A: sorted_int_array;
2.  c: int;
3.  compute i;
4.  fib1 := fib(i-3);
5.  fib2 := fib(i-2);
6.  m := fib2;
7.  repeat
8.    if c>A[m] then
9.      if fib1=0 then return false
10.     else
11.       m := m+fib1;
12.       tmp := fib1;
13.       fib1 := fib2-fib1;
14.       fib2 := tmp;
15.     end if;
16.   else if c<A[m]
17.     if fib2=1 then return false
18.     else
19.       m := m-fib1;
20.       fib2 := fib2 – fib1;
21.       fib1 := fib1 – fib2;
22.     end if;
23.   else return true;
24. until true;
```

Example

Search 3 in {1,2,3}

fib2	fib1	m
2	1	2
1	1	3

true

Search 6 in {1,2,3,4}

fib2	fib1	m
2	1	2
1	1	3
1	0	4

false

Search 100 in {1…10000}

fib2	fib1	m
4181	2584	4181
1597	987	1597
…	…	…

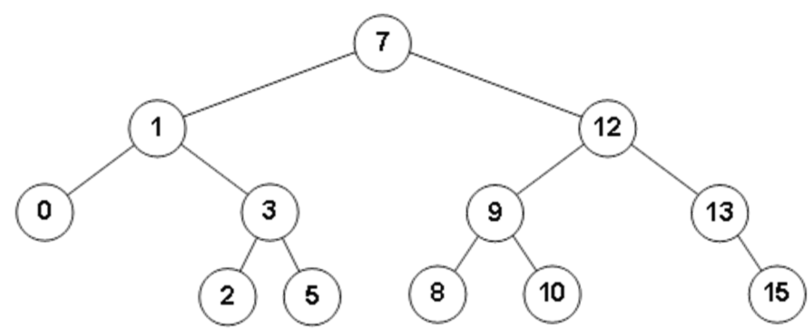
```
A: sorted_int_array;
c: int;
compute i;
fib1 := fib(i-3);
fib2 := fib(i-2);
m := fib2;
repeat
  if c>A[m] then
    if fib1=0 then return false
    else
      m := m+fib1;
      tmp := fib1;
      fib1 := fib2-fib1;
      fib2 := tmp;
    end if;
  else if c<A[m]
    if fib2=1 then return false
    else
      m := m-fib1;
      fib2 := fib2 – fib1;
      fib1 := fib1 – fib2;
    end if;
  else return true;
until true;
```

Complexity

- Worst-case: C is always in the larger (fib1) fraction of A
  - We roughly call once for n, once for 2n/3, once for 4n/9, …
- Formula of Moivre-Binet:
  - fib(i) = round$\left(\frac{\phi^i}{\sqrt{5}}\right) \approx \frac{\phi^i}{\sqrt{5}} \approx c * 1{,}62^i$
  - Where $\phi :=$ golden ratio ≈ 1.62
- We find fib such that fib(i-1)≤n≤fib(i)~c*$1{,}62^i$
- In worst-case, we make ~i comparisons
  - We break the array i times
- Since i=$\log_{1,62}$(n/c), we are in O(log(n))

Outlook: Searching without Math (later in this course)

- Will turn out that we actually can solve the search problem in O(log(n)) using only comparisons (no additions etc.)
- Transform A into a balanced binary search tree
  - At every node, the depth of the two subtrees differ by at most 1
  - At every node n, all values in the left (right) subtree are smaller (larger) than n
- Search
  - Recursively compare c to node labels and descend left/right
  - Balanced bin-tree has depth O(log(n))
  - We need at most log(n) comparisons – and nothing else



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Interpolation Search

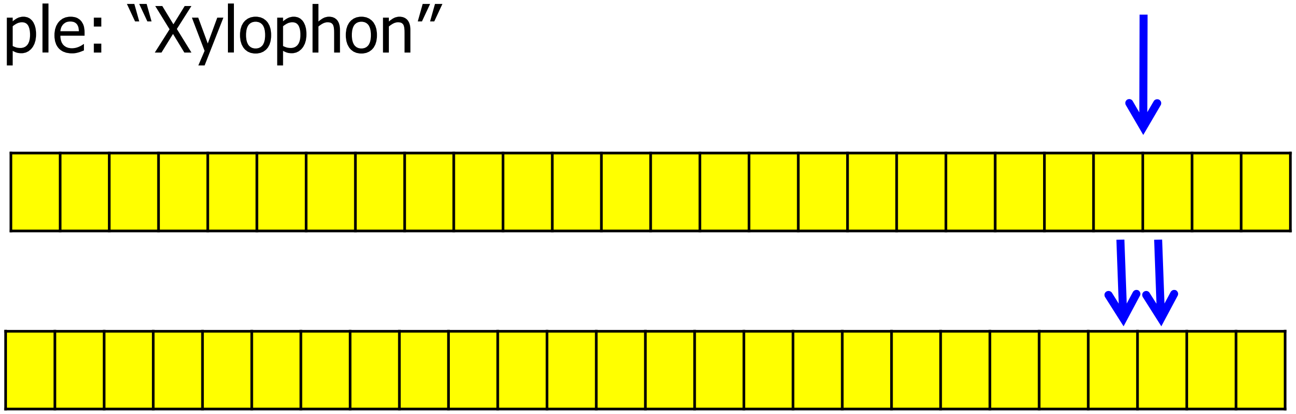
- Imagine you have a telephone book and search for „Zacharias“
- Will you open the book in the middle?
- We can exploit additional knowledge about our values
- Interpolation Search: Estimate where c lies in A based on the distribution of values in A
  - Simple: Use max and min values in A and assume equal distribution
  - Complex: Approximation of real distribution (histograms, …)

Simple Interpolation Search

- Assume equal distribution – values within A are equally distributed in range [A[1], A[n]]
- Best guess for the rank of c

$$rank(c) = l + (r - l) * \frac{c - A[l]}{A[r] - A[l]}$$

- Idea: Use m=rank(c) and proceed recursively
- Example: “Xylophon”



Analysis

- On average, Interpolation Search on equally distributed data requires O(log(log(n)) comparison (proof: see [OW])
- But: Worst-case is O(n)
  - If concrete distribution deviates heavily from expected distribution
  - E.g., A contains only names>"Xanthippe"
- Further disadvantage: In each phase, we perform ~4 adds/subs and 2*mults/divs
  - Assume this takes 12 cycles (1 mult/div = 4 cycles)
  - Binsearch requires 2*adds/subs + 1*div ~6 cycles
  - Even for $n=2^{32}$~4E9, this yields 12*log(log(4E9))~72 ops versus 6*log(4E9)~180 ops – not that much difference

Content of this Lecture

- Searching in Unsorted Lists
- Searching in Sorted Lists
- Selecting in Unsorted Lists
  - Naïve or clever

Quantiles

- The median of a list A is its middle value
  - Sort all values and take the one in the middle
- Generalization: q-quantiles
  - Sort all values and partition the list into subsequent bins of size q%*|A|
  - 25%, 50%, 75% are called quartiles
  - Median = 2-quantile

Selection Problem

- Definition
  The selection problem is to find the x%-quantile of a set A of unsorted values

- We can sort A and then access the quantile directly
- Thus, O(n*log(n)) is easy
- Can we solve this problem in linear time?
- It is easy to see that we have to look at least at each value once; thus, the problem is in $\Omega(n)$

Top-k Problem

- Top-k: Find the k largest values in A
- For constant k, a naïve solution is linear (and optimal)
  - repeat k times
  -   go through A and find largest value v;
  -   remove v from A;
  - return v
  - Requires k*|A|=O(|A|) comparisons
- But if k=x*|A|, we are in $O(x*|A|*|A|)=O(|A|^2)$
  - We measure complexity in size of the input
  - It is decisive whether k is part of the input or not

Selection Problem in Linear Time

- We sketch an algorithm which solves the problem for arbitrary x in linear time
  - Actually, we solve the equivalent problem of returning the k'th value in the sorted A (without sorting A)
- Interesting from a theoretical point-of-view
- Practically, the algorithm is of no importance because the linear factor gets enormously large
- It is instructive to see why (and where)

Algorithm

- Recall QuickSort: Chose pivot element p, divide array wrt p, recursively sort both partitions using the same trick
- We reuse the idea: Chose pivot element p, divide array wrt p, recursively select in the one partition that must contain the k'th element

```
func integer divide(A array;
                    l,r integer) {
  …
  while true
    repeat
      i := i+1;
    until A[i]>=val;
    repeat
      j := j-1;
    until A[j]<=val or j<i;
    if i>j then
      break while;
    end if;
    swap( A[i], A[j]);
  end while;
  swap( A[i], A[r]);
  return i;
}
```

```
func int quantile(A array;
                  k, l, r int) {
  if r≤l then
    return A[l];
  end if;
  pos := divide( A, l, r);
  if (k ≤ pos-1) then
    return quantile(A, k, l, pos-1);
  else
    return quantile(A, k-pos+1, pos, r);
  end if;
}
```

Analysis

```
func int quantile(A array;
                       k, l, r int) {
  if r≤l then
    return A[l];
  end if;
  pos := divide( A, l, r);
  if (k ≤ pos-l) then
    return quantile(A, k, l, pos-1);
  else
    return quantile(A, k-pos+l, pos, r);
  end if;
}
```

- Worst-case: Assume arbitrarily badly chosen pivot elements
- pos always is r-1 (or l+1)
- Gives $O(n^2)$
- Need to chose the pivot element p more carefully

Choosing p

- Assume we can chose p such that we always continue with at most $q=y\%$ of A
  - "y%" means: Extend of reduction depends on n
- We perform at most $T(n) = T(q*n) + c*n$ comparisons
  - $T(q*n)$ – recursive descent
  - $c*n$ – function "divide"
- $T(n) = T(q*n)+c*n = T(q^2*n)+q*c*n+c*n = T(q^2n)+(q+1)*c*n = T(q^3n)+(q^2+q+1)*c*n = \ldots$

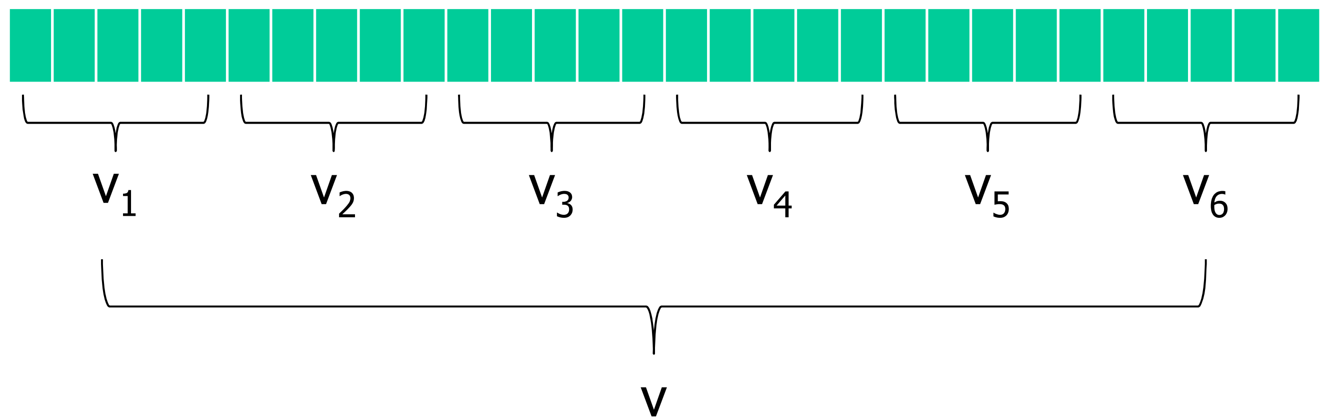
$$\underset{n\to\infty}{T(n)} = c*n*\sum_{i=0}^{n} q^i \le c*n*\sum_{i=0}^{\infty} q^i = c*n*\frac{1}{1-q} = O(n)$$

Discussion

- Our algorithm has worst-case complexity O(n) when we manage to always reduce the array by a fraction of its size – no matter, how large the fraction
- This is not an average-case. We must always (not on average) cut some fraction of A
- Eh – magic?
- No – follows from the way we defined complexity and what we consider as input
- Many ops are "hidden" in the linear factor
  - q=0.9: c*10*n
  - q=0.99: c*100*n
  - q=0.999: c*1000*n

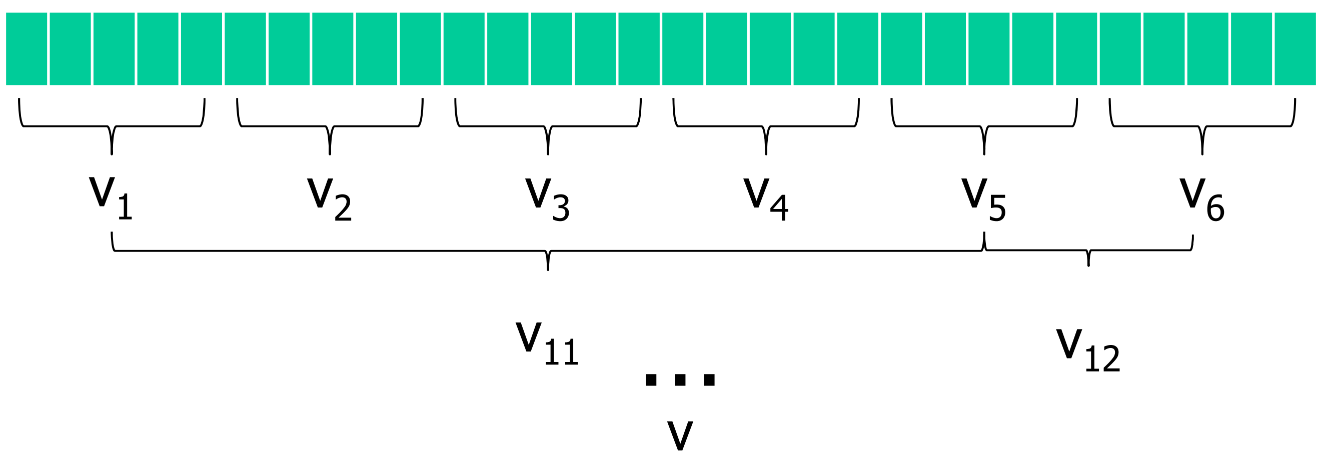
Median-of-Median (Assume $|A| = 5^l$)

- How can we guarantee to always cut a fraction of A?
- Median-of-median algorithm
  - Partition A in stretches of length 5
  - Compute the median v_i for each partition
  - Use the (approximated) median v of all v_i as pivot element



Complexity

- Run through A in jumps of length 5
- Find each median in constant time
  - Runtime of sorting a list of length 5 does not depend on n
- Call algorithm recursively on all medians
- Since we always reduce the range of values to look at by 80%, this requires O(n) time



Why Does this Help?

- We have n/5 first-level medians v_i
- v (as median of medians) is smaller than halve of them and greater than the other half (both are n/10 values)
- Each v_i itself is smaller than (and greater than) 2 values
- Thus v is smaller than (and greater than) at least 3*n/10 elements

Illustration (source: Wikipedia)

	12	15	11	2	9	5	0	7	3	21	44	40	1	18	20	32	19	35	37	39
	13	16	14	8	10	26	6	33	4	27	49	46	52	25	51	34	43	56	72	79
Median	17	23	24	28	29	30	31	36	42	47	50	55	58	60	63	65	66	67	81	83
	22	45	38	53	61	41	62	82	54	48	59	57	71	78	64	80	70	76	85	87
	96	95	94	86	89	69	68	97	73	92	74	88	99	84	75	90	77	93	98	91

- Median-of-median of a randomly permuted list 0..99
- For clarity, each 5-tuple is sorted (top-down) and all 5-tuples are sorted by median (left-right)
- Gray/white: Values with actually smaller/greater than med-of-med 47
- Blue: Range with certainly smaller / larger values