Algorithms and Data Structures

Open Hashing

Marius Kloft

RECAP: BLOOM FILTER

Searching an Element

- Assume we want to know if k is an element of a list S of 32bit integers – but S is very large
  - We shall from now on count in "keys" = 32bit
- S must be stored on disk
  - Assume testing k in memory costs very little, but loading a block (size b=1000 keys) from disk costs enormously more
  - Thus, we only count IO – how many blocks do we need to load?
- Assume |S|=1E9 (1E6 blocks) and we have enough memory for 1E6 keys
  - Thus, enough for 1000 of the 1 Million blocks

Options

- If S is not sorted
  - If $k \in S$, we need to load 50% of S on average: ~ 0.5E6 IO
  - If $k \notin S$, we need to load S entirely: ~ 1E6 IO
- If S is sorted
  - It doesn't matter whether $k \in S$ or not
  - We need to load log(|S|/b)=log(1E6)~20 blocks
- Notice that we are not using our memory …

Idea of a Bloom Filter

- Build a hash map A as big as the memory
- Use A to indicate whether a key is in S or not
- The test may fail, but only in one direction
  - If $k \in A$, we don't know for sure if $k \in S$
  - If $k \notin A$, we know for sure that $k \notin S$
- A acts as a filter: A Bloom filter
  - Bloom, B. H. (1970). "Space/Time Trade-offs in Hash Coding with Allowable Errors." Communications of the ACM 13(7): 422-426.

Bloom Filter: Simple

- Create a bitarray A with |A|=a=1E6*32
  - We fully exploit our memory
  - A is always kept in memory
- Choose a uniform hash function h
- Initialize A (offline): ∀k∈S: A[h(k)]=1
- Searching k given A (online)
  - Test A[h(k)] in memory
  - If A[h(k)]=0, we know that k∉S (with 0 IO)
  - If A[h(k)]=1, we need to search k in S

Bloom Filter: Advanced

- Create a bitarray A with |A|=a=1E6*32
  - We fully exploit our memory
  - A is always kept in memory
- Choose j independent uniform hash functions h_j
  - Independent: The values of one hash function are statistically independent of the values of all other hash functions
- Initialize A (offline): $\forall k \in S$, $\forall j$: $A[h_j(k)]=1$
- Searching k given A (online)
  - $\forall j$: Test $A[h_j(k)]$ in memory
  - If any of the $A[h_j(k)]=0$, we know that $k \notin S$
  - If all $A[h_j(k)]=1$, we need to search k in S

Analysis

- Assume $k \notin S$
  - Let denote C_n the cost of such a (negative) search
  - We only access disk if all $A[h_j(k)]=1$ by chance – how often?
  - In all other cases, we perform no IO and assume 0 cost
- Assume $k \in S$
  - We will certainly access disk, as all $A[h_j(k)]=1$ but we don't know if this is by chance of not
  - Thus, $C_p = 20$
    - Using binsearch, assuming S is kept sorted on disk

Chances for a False Positive

- For one $k \in S$ and one hash function, the chance for a given position in A to be 0 is $1-1/a$
- For j hash functions, chance that all remain 0 is $(1-1/a)^j$
- For j hash functions and n values, the chance to remain 0 is $q=(1-1/a)^{j*n}$
- Prob. of a given bit being 1 after inserting n values is $1-q$
- Now let's look at a search for key k, which tests j bits
- Chance that all of these are 1 by chance is $(1-q)^j$
  - By chance means: Case when k is not in S
- Thus, $C_n=(1-q)^j*C_p + (1-(1-q)^j)*0$
  - In our case, for j=5: 0.001; j=10: 0.000027

Average Case

- Assume we look for all possible values ($|U|=u=2^{32}$) with the same probability
- (u-n)/u of the searches are negative, n/u are positive
- Average cost per search is

$$c_{avg} := ((u-n)*C_n + n*C_p) / u$$

- For j=5: 0,14
- For j=10: 0,13
  - Larger j decreases average cost, but increase effort for each single test
  - What is the optimal value for j?
- Much better than sorted lists

OPEN HASHING

Open Hashing

- Open Hashing: Store all values inside hash table A
- Inserting values
  - No collision: Business as usual
  - Collision: Chose another index and probe again (is it "open"?)
  - As second index might be full as well, probing must be iterated
- Many suggestions on how to chose the next index to probe
- In general, we want a strategy (probe sequence) that
  - … ultimately visits any index in A (and few twice before)
  - … is deterministic – when searching, we must follow the same order of indexes (probe sequence) as for inserts

Reaching all Indexes of A

- Definition
  Let A be a hash table, |A|=m, over universe U and h a hash function for U into A. Let I={0, …, m-1}. A probe sequence is a deterministic, surjective function s: UxI→I
- Remarks
  - We use j to denote elements of the sequence: Where to jump after j-1 probes
  - s need not be injective – a probe sequences may cross itself
    - But it is better if it doesn't
  - We typically use s(k, j) = (h(k) – s'(k, j)) mod m for a properly chosen function s'
- Example: s'(k, j) = j ,hence s(k, j) = (h(k)–j) mod m

Searching

```
func int search(k int) {
  j := 0;
  first := h(k);
  repeat
    pos := (first-s’(k, j) mod m;
    j := j+1;
  until (A[pos]=k) or
        (A[pos]=null) or
        (j=m);
  if (A[pos]=k) then
    return pos;
  else
    return -1;
  end if;
}
```

- Let s’(k, 0) := 0
- We assume that s cycles through all indexes of A
  - In whatever order
- Probe sequences longer than m-1 usually make no sense, as they necessarily look into indexes twice
  - But beware of non-injective functions

Deletions

- Deletions are a problem
  - Assume h(k)= k mod 11 and s(k, j) = (h(k) + 3*j) mod m)

	0	1	2	3	4	5	6	7	8	9	10
ins(1); ins(6)		**1**					**6**				
ins(23)		**1**			**23**		**6**				
ins(12)		**1**			**23**		**6**	**12**			
del(23)		**1**					**6**	**12**			
search(12)		**1**			**?**		**6**	**12**			

Remedies

- Leave a mark (tombstone)
  - During search, jump over tombstones
  - During insert, tombstones may be replaced
- Re-organize list
  - Keep pointer p to index where a key should be deleted
  - Walk to end of probe sequence (first empty entry)
  - Move last non-empty entry to index p
  - Requires to run through the probe entire sequence for every deletion (otherwise only n/2 on average)
  - Not compatible with strategies that keep probe sequences sorted
    - See later

Open versus External collision handling

- Pro
  - We do not need more space than reserved – more predictable
  - A typically is filled more homogeneously – less wasted space
- Contra
  - More complicated
  - Generally, we get worse WC/AC complexities for insertion/deletion
    - Additional work to run down probe sequences
    - Especially deletions have overhead
  - A gets full; we cannot go beyond α=1

Open Hashing: Overview

- We will look into three strategies
  - Linear probing: s(k, j) := (h(k) – j) mod m
  - Double hashing: s(k, j) := (h(k) – j*h'(k)) mod m
  - Ordered hashing: Any s; values in probe sequence are kept sorted
- Others
  - Quadratic hashing: s(k, j) := $(h(k) - \text{floor}(j/2)^2 * (-1)^j)$ mod m
    - Less vulnerable to local clustering then linear hashing
  - Uniform hashing: s is a random permutation of I dependent on k
    - High administration overhead, guarantees shortest probe sequences
  - Coalesced hashing: s arbitrary; entries are linked by add. pointers
    - Like overflow hashing, but overflow chains are in A; needs additional space for links

Content of this Lecture

- Open Hashing
  - Linear Probing
  - Double Hashing
  - Ordered Hashing

Linear Probing

- Probe sequence function: s(k, j) := (h(k) – j) mod m
  - Assume h(k)= k mod 11

	0	1	2	3	4	5	6	7	8	9	10
ins(1); ins(7); ins(13)		1	13					7			
ins(23)	23	1	13					7			
ins(12)	23	1	13					7			12
ins(10)	23	1	13					7		10	12
ins(24)	23	1	13					7	24	10	12

Analysis

- The longer a chain …
  - the more different values of h(k) it covers
  - the higher the chances to produce more collisions
- The faster it grows, the faster it merges with other chains
- Assume an empty position p left of a chain of length n and an empty position q with an empty cell to the right
  - Also assume h is uniform
  - Chances to fill q with next insert: 1/m
  - Chances to fill p with the next insert: (n+1)/m
- Linear probing tends to quickly produce long, completely filled stretches of A with high collision probabilities

In Numbers (Derivation of Formulas Skipped)

- Scenario: Some inserts, then many searches
  - Expected number of probes per search are most important

erfolgreiche Suche:

$$C_n \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)}\right)$$

erfolglose Suche:

$$C'_n \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

α	C_n (erfolgreich)	C'_n(erfolglos)
0.50	1.5	2.5
0.90	5.5	50.5
0.95	10.5	200.5
1.00	–	–

Source: S. Albers / [OW93]

Quadratic Hashing

erfolgreiche Suche:

$$C_n \approx 1 - \frac{\alpha}{2} + \ln\left(\frac{1}{(1-\alpha)}\right)$$

erfolglose Suche:

$$C'_n \approx \frac{1}{1-\alpha} - \alpha + \ln\left(\frac{1}{(1-\alpha)}\right)$$

α	C_n (erfolgreich)	C'_n (erfolglos)
0.50	1.44	2.19
0.90	2.85	11.40
0.95	3.52	22.05
1.00	-	-

Source: S. Albers / [OW93]

Discussion

- Disadvantage of linear (and quadratic) hashing:
  Problems with the original hash function h are preserved
  - Probe sequence only depends on h(k), not on k
    - s'(k, j) ignores k
  - All synonyms k, k' will create the same probe sequence
    - Two keys that form a collision are called synonyms
  - Thus, if h tends to generate clusters (or inserted keys are non-uniformly distributed in U), also s tends to generate clusters (i.e., sequences filled from multiple keys)

Content of this Lecture

- Open Hashing
  - Linear Probing
  - Double Hashing
  - Ordered Hashing

Double Hashing

- Double Hashing: Use a second hash function h'
  - s(k, j) := (h(k) − j*h'(k)) mod m (with h'(k)≠0)
  - Further, we don't want that h'(k)|m (done if m is prime)
- h' should spread h-synonyms
  - If h(k)=h(k'), then hopefully h'(k)≠h'(k')
    - Otherwise, we preserve problems with h
  - Optimal case: h' statistically independent of h, i.e.,

    $p(h(k)=h(k') \wedge h'(k)=h'(k')) = p(h(k)=h(k'))*p(h'(k)=h'(k'))$

    - If both are uniform: $p(h(k)=h(k')) = p(h'(k)=h'(k')) = 1/m$
- Example: If h(k)= k mod m, then h'(k)=1+k mod (m-2)

Example (Linear Probing produced 9 collisions)

$h(k) = k \bmod 11$; $h'(k) = 1 + k \bmod 9$; $s(k,j) := (h(k) - j \cdot h'(k)) \bmod 11$

	0	1	2	3	4	5	6	7	8	9	10
ins(1); ins(7); ins(13)		1	13					7			
ins(23) h(k)=1; h'(k)=6 s(k, 1)=6		1	13				23	7			
ins(12) h(k)=1; h'(k)=4 s(k, 1)=8		1	13				23	7	12		
ins(10)		1	13				23	7	12		10
ins(24) h(k)=2; h'(k)=7 s(k, 1)=6 s(k, 2)=10 s(k, 3)=3		1	13	24			23	7	12		10

Analysis

- Please see [OW93]

$$C'_n \leq \frac{1}{1 - \alpha}$$

$$C_n \approx \frac{1}{\alpha} * \ln\left(\frac{1}{(1 - \alpha)}\right)$$

α	C_n (erfolgreich)	C'_n(erfolglos)
0.50	1.39	2
0.90	2.56	10
0.95	3.15	20
1.00	-	-

Another Example

	0	1	2	3	4	5	6	7	8	9	10
ins(23); ins(13)		**23**	**13**								
ins(34) h(k)=1; h‘(k)=8 s(k, 1)=4		**23**	**13**		**34**						
ins(12) h(k)=1; h‘(k)=4 s(k, 1)=8		**23**	**13**		**34**				**12**		
ins(10)		**23**	**13**		**34**				**12**		**10**
ins(15) h(k)=4; h‘(k)=7 s(k, 1)=8 s(k, 2)=1 s(k,3)=5		**23**	**13**		**34**	**15**			**12**		**10**

Observation

- We change the order of insertions (and nothing else)

	0	1	2	3	4	5	6	7	8	9	10
ins(23); ins(13)		**23**	**13**								
ins(15) h(k)=4; h‘(k)=6		**23**	**13**		**15**						
ins(12) h(k)=1; h‘(k)=4 s(k, 1)=8		**23**	**13**		**15**				**12**		
ins(10)		**23**	**13**		**15**				**12**		**10**
ins(34) h(k)=1; h‘(k)=8 s(k, 1)=4 s(k, 2)=7		**23**	**13**		**15**			**34**	**12**		**10**

Observation

- The number of collisions depends on the order of inserts
  - Because h' spreads h-synonyms differently for different values of k
- We cannot change the order of inserts, but …
- Observe that when we insert k' and there already was a k with h(k)=h(k'), we actually have two choices
  - Until now we always looked for a new place for k'
  - Why not: set A[h(k')]=k' and find a new place for k?
  - If s(k',1) is filled but s(k,1) is free, then the second choice is better
  - Insert is faster, searches will be faster on average

Brent’s Algorithm

Brent, R. P. (1973). "Reducing the Retrieval Time of Scatter Storage Techniques." CACM

- Brent’s algorithm:
  Upon collision, propagate key for which the next index in probe sequence is free; if both are occupied, propagate k’
- Improves only successful searches
  - Otherwise we have to follow the chain to its end anyway
- One can show that the average-case probe length for successful searches now is constant (~2.5 accesses)
  - Even for relatively full tables

Content of this Lecture

- Open Hashing
  - Linear Probing
  - Double Hashing
  - Ordered Hashing

Idea

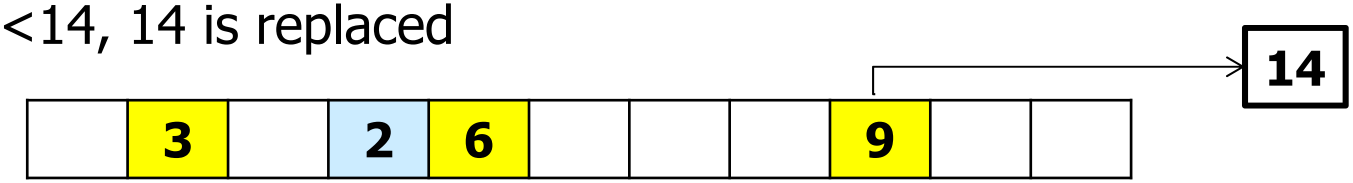
- Can we do something to improve unsuccessful searches?
  - Recall overflow hashing: If we keep the overflow chain sorted, we can stop searching after $\alpha/2$ comparisons on average
- Transferring this idea: Keep keys sorted in any probe seq.
  - We have seen with Brent's algorithm that we have the choice which key to propagate whenever we have a collision
  - Thus, we can also choose to always propagate the larger of both keys – which generates a sorted probe sequence
- Result: Unsuccessful are as fast as successful searches

Details

- In Brent‘s algorithm, we only replace a key if we can insert the replaced key directly into A
- Now, we must replace keys even if the next slot in the probe sequence is occupied
  - We run through probe sequence until we meet a key that is larger
  - We insert the new key here
  - All subsequent keys must be replaced (moved in probe sequence)
- Note that this doesn’t make inserts slower than before
  - Without replacement, we would have to search the first free slot
  - Now we replace until the first free slot

Critical Issue

- Imagine ins(6) would first probe position 1, then 4
- Since 6<9, 9 is replaced; imagine the next slot would be 8
- Since 9<14, 14 is replaced



- Problem
  - 14 is not a synonym of 9 – two probe sequences cross each other
  - Thus, we don't know where to move 14 – the next position in general requires to know the "j", i.e., the number of hops that were necessary to get from h(14) to slot 8
- Ordered hashing only works if we can compute the next offset without knowing j
  - E.g. linear hashing (offset -1) or double hashing (offset –h'(k))

Correctness

- Invariant: Let s(k,j) be the position in A where k is stored. Searching k returns the correct answer iff $\forall i<j$: $A[s(k,i)] < A[s(k,j)]$
- Proof by induction
  - Invariant holds for the empty array
  - Imagine invariant holds before inserting a key k'
  - We insert k' in position s(k',j) (for some j)
    - Either A[s(k',j)] was free
      - then invariant still holds
    - Or the old A[s(k',j)]>k' (otherwise we wouldn't have inserted k' here)
      - Then the old A[s(k',j)] was replaced by a smaller value
      - Invariant must still hold

Wrap-Up

- Open hashing can be a good alternative to overflow hashing even if the fill grade approaches 1
  - Very little average-case cost for look-ups with double hashing and Brent's algorithm or using ordered hashing
    - Depending which types of searches are more frequent
- Open hashing suffers from having only static place, but guarantees to not request more space once A is allocated
  - Less memory fragmentation

Exemplary Questions

- Create a hashtable step-by-step using open hashing with double probing and hash functions h(k)=k mod 13 and h'(k)=3+k mod 9 when inserting keys 17,12,4,1,36,25,6
- Use the same list for creating a hash table with double hashing and Brent's algorithm
- Use the same list for creating a hash table with ordered linear probing (linear probing such that the probe sequences are ordered).
- Analyze the WC complexity of searching key k in a hash table with direct chaining using a sorted linked list when (a) k is in A; (b) k is not in A.